Short Course

State Space Models, Generalized Dynamic Systems
and
Sequential Monte Carlo Methods,
and
their applications
in Engineering, Bioinformatics and Finance

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1.4 Introduction Sequential Monte Carlo Methods

$$x_t \sim q_t(\cdot \mid x_{t-1}) \qquad\qquad y_t \sim f_t(\cdot \mid x_t)$$

Let $\boldsymbol{y}_t = (y_1, \ldots, y_t)$**.**

t	$t+1$
$p(x_t \mid \boldsymbol{y}_t)$	$p(x_{t+1} \mid \boldsymbol{y}_t, y_{t+1})$

$$\begin{array}{ccc} x_t^{(1)} & \longrightarrow & x_{t+1}^{(1)} \\ \vdots & & \vdots \\ x_t^{(m)} & \longrightarrow & x_{t+1}^{(m)} \end{array}$$

Note that

$$p(x_t, x_{t+1} \mid y_1, \ldots, y_t) \propto q_t(x_{t+1} \mid x_t)p(x_t \mid y_1, \ldots, y_t)$$

$$\begin{array}{ccc} t & t+\frac{1}{2} & t+1 \\ \hline p(x_t \mid \boldsymbol{y}_t) & p(x_{t+1} \mid \boldsymbol{y}_t) & p(x_{t+1} \mid \boldsymbol{y}_t, y_{t+1}) \\ \searrow & \nearrow & \\ & q_t(\cdot \mid x_t) & \end{array}$$

$$\begin{array}{ccccc} x_t^{(1)} & \longrightarrow & x_{t+1}^{(1)} & \longrightarrow & \textbf{?} \\ \vdots & & \vdots & & \vdots \\ x_t^{(m)} & \longrightarrow & x_{t+1}^{(m)} & \longrightarrow & \textbf{?} \end{array}$$

Note that

$$p(x_t, x_{t+1} \mid y_1, \ldots, y_t, y_{t+1}) \propto f_t(y_{t+1} \mid x_{t+1}) p(x_t, x_{t+1} \mid y_1, \ldots, y_t)$$

$$\begin{array}{ccccc}
t & & t + \frac{1}{2} & & t+1 \\
\hline
p(x_t \mid \boldsymbol{y}_t) & & p(x_{t+1} \mid \boldsymbol{y}_t) & & p(x_{t+1} \mid \boldsymbol{y}_t, y_{t+1}) \\
& \searrow \qquad \nearrow & & \searrow \qquad \nearrow & \\
& q_t(\cdot \mid x_t) & & w_{t+1} = f_{t+1}(y_{t+1} | x_{t+1}) &
\end{array}$$

$$\cdots\cdots\cdots \quad \cdots \quad \cdots\cdots\cdots\cdots \quad \cdots \quad \cdots\cdots\cdots\cdots\cdots$$

$$\begin{array}{ccccc}
x_t^{(1)} & \longrightarrow & x_{t+1}^{(1)} & \longrightarrow & (x_{t+1}^{(1)}, w_{t+1}^{(1)}) \\
\vdots & & \vdots & & \vdots \\
x_t^{(m)} & \longrightarrow & x_{t+1}^{(m)} & \longrightarrow & (x_{t+1}^{(m)}, w_{t+1}^{(m)})
\end{array}$$

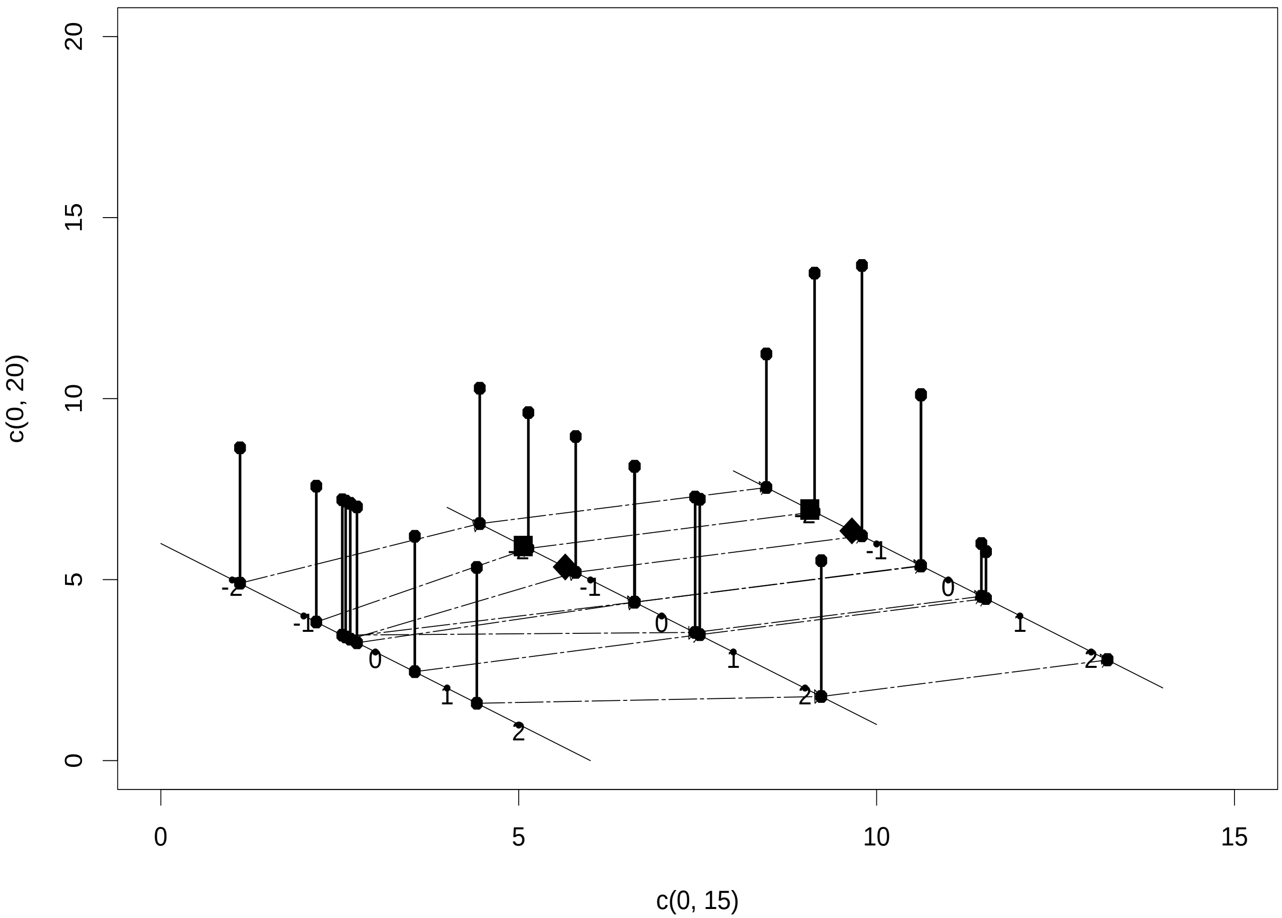
Example:

$$x_t = x_{t-1} + e_t$$

$$y_t = x_t + \varepsilon_t$$

where $e_t \sim N(0,1)$**,** $\varepsilon_t \sim N(0,1)$**, and**

$$x_0 \sim N(0,1)$$



c(0, 20)
c(0, 15)
0
5
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15
20
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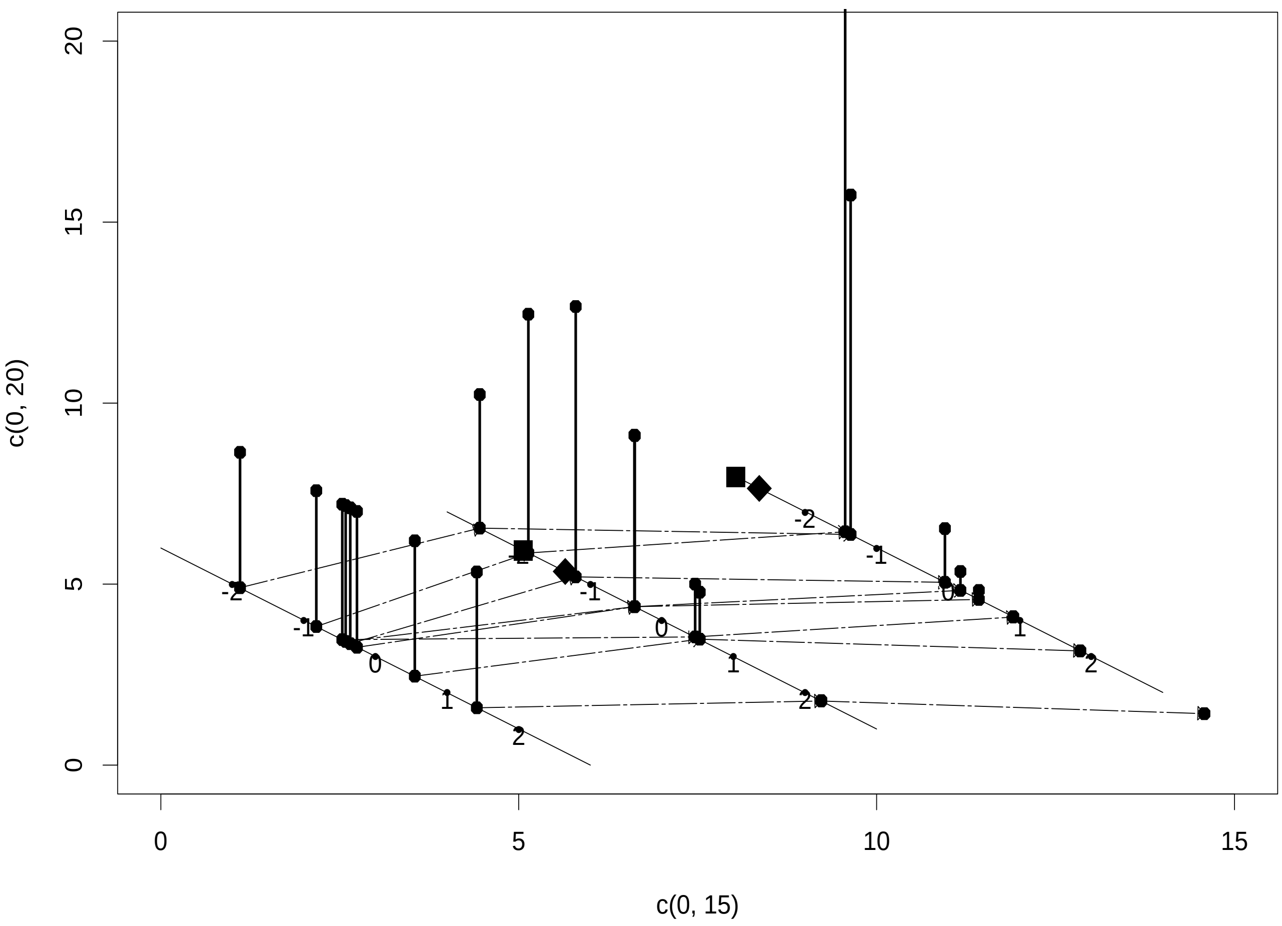
2



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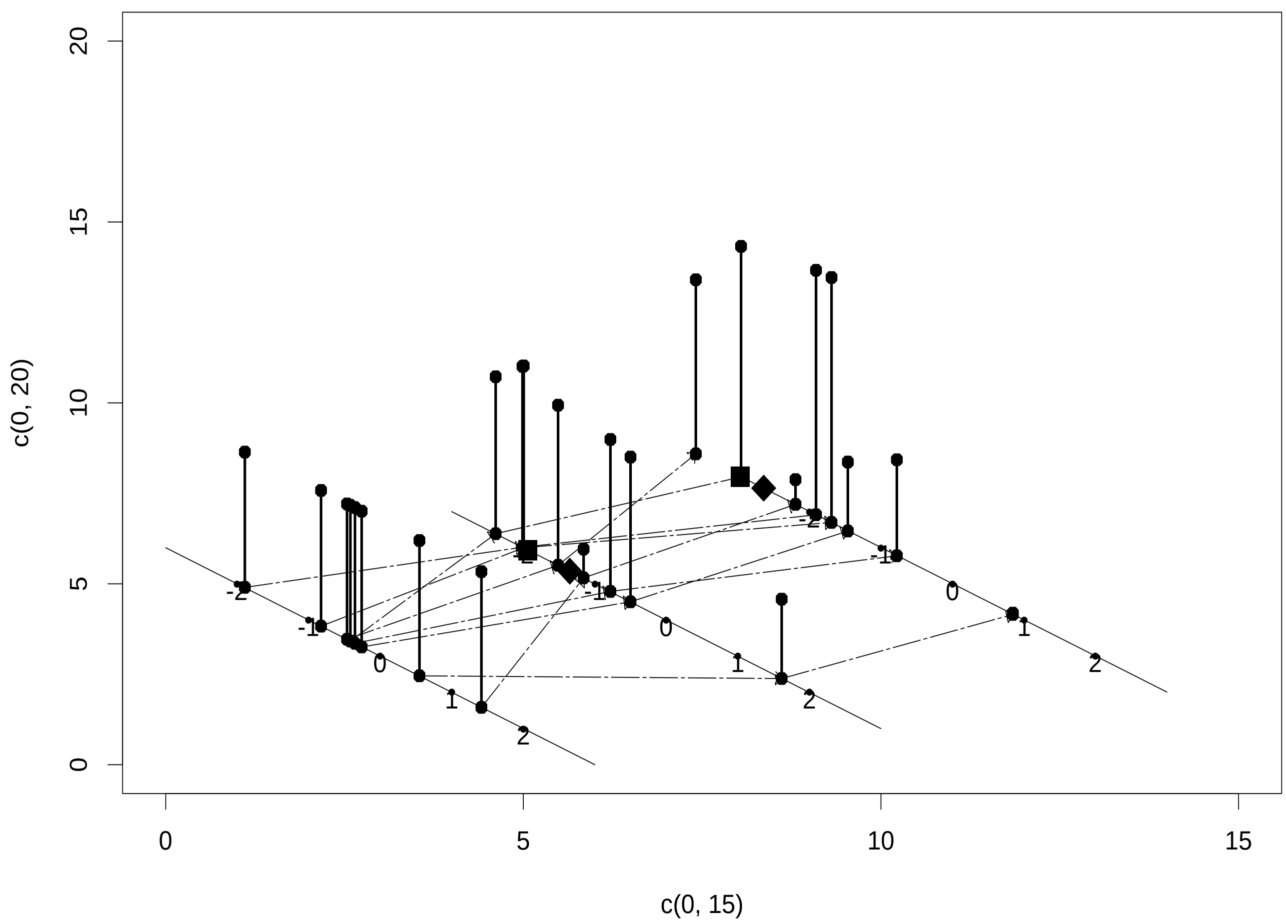

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Sequential Importance Sampling

SIS Step: for $j = 1, \ldots, m$:

(A) Draw $x_t^{(j)}$ from $g(x_t \mid x_{t-1}^{(j)}, y_t)$

(B) Compute the incremental weight

$$u_t^{(j)} = \frac{p(\boldsymbol{x}_t^{(j)} \mid \boldsymbol{y}_t)}{p_{t-1}(\boldsymbol{x}_{t-1}^{(j)} \mid \boldsymbol{y}_{t-1}) g(x_t^{(j)} \mid x_{t-1}^{(j)})}$$

and the new weight

$$w_t^{(j)} = u_t^{(j)} w_{t-1}^{(j)}$$

Proof:

$$
\begin{aligned}
w_t^{(j)} &= \prod_{s=1}^{t} u_s^{(j)} \\
&= \frac{p(\boldsymbol{x}_t^{(j)} \mid \boldsymbol{y}_t)}{p_{t-1}(\boldsymbol{x}_{t-1}^{(j)} \mid \boldsymbol{y}_{t-1}) g(x_t^{(j)} \mid x_{t-1}^{(j)})} \times \frac{p(\boldsymbol{x}_{t-1}^{(j)} \mid \boldsymbol{y}_{t-1})}{p_{t-2}(\boldsymbol{x}_{t-2}^{(j)} \mid \boldsymbol{y}_{t-2}) g(x_{t-1}^{(j)} \mid x_{t-2}^{(j)})} \cdots \\
&\quad \times \cdots \times \frac{p(\boldsymbol{x}_2^{(j)} \mid \boldsymbol{y}_2)}{p_1(\boldsymbol{x}_1^{(j)} \mid \boldsymbol{y}_1) g(x_2^{(j)} \mid x_1^{(j)})} \times \frac{p(\boldsymbol{x}_1^{(j)} \mid \boldsymbol{y}_1)}{g(x_1^{(j)})} \\
&= \frac{p(\boldsymbol{x}_t^{(j)} \mid \boldsymbol{y}_t)}{g_1(x_1) \prod_{s=2}^{t} g(x_s^{(j)} \mid x_{s-1}^{(j)})}
\end{aligned}
$$


Are we done?